# Behavior of Turbulent Flow near a Porous Wall with Pressure Gradient

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**Van Driest's theory, which provides a continuous velocity and shear distribution for turbulent flow near a nonporous wall, is extended to turbulent flow near a porous wall. The new, modified theory enables the theoretical calculation of velocity profiles to be performed for a wider range of mass-transfer rates, and it gives good agreement with experimental data.**

## Nomenclature

$c_f$ = local skin-friction coefficient, $\tau_w/(\frac{1}{2})\rho u_e^2$
$l$ = mixing length
$p$ = pressure
$p^+$ = $-(dp/dx)[\nu/\rho(u_w^*)^3]$
$u,v$ = $x$ and $y$ components of velocity, respectively
$u_w^*$ = friction velocity, $(\tau_w/\rho)^{1/2}$
$x,y$ = rectangular coordinates
$y^+$ = $yu_w^*/\nu$
$\delta$ = boundary-layer thickness
$\theta$ = momentum thickness
$\mu$ = dynamic viscosity
$\nu$ = kinematic viscosity
$\rho$ = density
$\tau$ = shear stress

***Subscripts***

$e$ = boundary-layer edge
$i$ = inner region
$o$ = outer region
$t$ = turbulent
$w$ = wall

## 1. Introduction

In Ref. 1, Van Driest introduced a very useful modification of Prandtl's mixing length theory, which provided a continuous velocity and shear distribution for flat-plate turbulent flow near a nonporous wall. This modification also formed the basis for the theoretical calculation of the velocity profiles and has been used quite successfully by several investigators, for example by Cebeci and Smith[2] and Patankar and Spalding.[3]

The purpose of this paper is to show a possible method of extending Van Driest's modification to turbulent flow near a porous wall with pressure gradient. The new modification enables the theoretical calculation of velocity profiles to be performed for a wide range of mass-transfer rates and gives good agreement with experimental data.

## 2. Analysis

### 2.1 Van Driest's Analysis

We consider Stokes' flow, that is, a flow about an infinite flat plate that executes oscillations parallel to itself. The governing momentum equation for the flow is the one-dimensional nonsteady momentum equation. For an incompressible flow it is given by

$$\partial u/\partial t = \nu \partial^2 u/\partial y^2 \qquad (1)$$

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If we assume that the wall velocity is

$$u(o,t) = u_0 \cos\omega t \qquad (2)$$

the solution of Eq. (1) subject to the boundary condition Eq. (2) is given by the following expression:

$$u = u_0 e^{-ny} \cos(\omega t - ny) \qquad (3)$$

where

$$n = (\omega/2\nu)^{1/2} \qquad (4)$$

From Eq. (3), we see that the amplitude of the motion diminishes with distance from the wall as a consequence of the factor $\exp(-ny)$. If we identify $u$ as the fluctuation velocity $u'$, we see that when the plate is fixed and the fluid oscillates relative to the plate, the fluctuation velocity (with the cosine term neglected) will be

$$u' = u_0'(1 - e^{-ny}) \qquad (5)$$

Equation (5) shows that because of the viscous effects it is necessary to correct the velocity fluctuation by $1 - \exp(-ny)$. From the definition of Reynolds shear stress, we then can write

$$-\rho\langle u'v'\rangle = -\rho\langle u_0'v_0'\rangle(1 - e^{-ny})^2 \qquad (6)$$

Since according to Prandtl's mixing length concept,

$$-\langle u_0'\nu_0'\rangle = l^2(\partial u/\partial y)^2 \qquad (7)$$

we see that Eq. (6) can now be written as

$$-\rho\langle u'v'\rangle = l^2(1 - e^{-ny})^2(\partial u/\partial y)^2 \qquad (8)$$

where $l = ky$.

The exponential term in Eq. (3) can be written as

$$\exp(-ny) = \exp -(\omega/2\nu)^{1/2}y = \exp(-y/A_1) \qquad (9)$$

where

$$A_1 = 2^{1/2}\nu(1/\omega\nu)^{1/2} \qquad (10)$$

The units of $(\omega\nu)^{1/2}$ are that of a velocity. Van Driest takes this characteristic velocity to be the friction velocity, $u_w^* = (\tau_w/\rho)^{1/2}$, writes Eq. (10) as

$$A = \text{const}\,\nu(\tau_w/\rho)^{-1/2} \qquad (11)$$

and empirically determines the constant to be 26 for $k = 0.4$. Then it follows from Eq. (8) that the Reynolds shear stress term can be written as

$$-\rho\langle u'v'\rangle = (0.4y)^2[1 - \exp(-y/A)]^2(\partial u/\partial y)^2 \qquad (12)$$

The exponential term can also be written as $\exp -(y^+/A^+)$ where

$$y^+ = yu_w^*/\nu,\ A^+ = 26 \qquad (13)$$

### 2.2 Present Analysis

The expression given by Eq. (11) was obtained for a flat-plate flow with no mass transfer. As it stands, it cannot be used for flows with pressure gradients. This is quite obvious, since for a flow with an adverse pressure gradient $\tau_w \to 0$, which introduces a discontinuity in shear stress and in velocity profiles. It can be extended to flows with pressure gradient and mass transfer by the following reasoning.

Consider the momentum equation for two-dimensional incompressible turbulent flows:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} - \rho\langle u'v'\rangle\right) \quad (14)$$

In the laminar sublayer region this equation can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \quad (15)$$

by neglecting the Reynolds shear-stress term $-\rho\langle u'v'\rangle$, and by putting $\tau = \mu\partial u/\partial y$. If we identify the characteristic velocity in Eq. (10) to be the friction velocity based on the sublayer thickness rather than its wall friction velocity, then Eq. (11) can be written as

$$A = 26\nu(\tau_s/\rho_s)^{-1/2} \quad (16)$$

where the subscript denotes sublayer.

If we approximate Eq. (15) by the following equation

$$d\tau_s/dy - (v_w/\nu)\tau_s = dp/dx \quad (17)$$

The solution of Eq. (17) with $y_s^+ = 11.8$ enables the damping constant $A^+$ to be written as†

$$A^+ = 26\left\{-\frac{p^+}{v_w^+}\left[\exp(11.8\, v_w^+) - 1\right] + \exp(11.8\, v_w^+)\right\}^{-1/2} \quad (18)$$

Here

$$p^+ = -(dp/dx)\nu/\rho(u_w^*)^3,\; v_w^+ = v_w/u^* \quad (19)$$

For a nonporous wall with pressure gradient, Eq. (18) reduces to

$$A^+ = 26[1 - 11.8p^+]^{-1/2} \quad (20)$$

For a porous flat-plate flow, Eq. (18) reduces to

$$A^+ = 26\exp(-5.9\, v_w^+) \quad (21)$$

Figure 1 shows the deviation of the damping constant from that of a flat-plate flow with no mass transfer, that is, $A^+/26$. Figure 1a shows a plot of Eq. (20). According to this plot, the damping constant of a pressure-gradient flow deviates considerably from that of a flat-plate flow and becomes quite large as $p^+$ approaches the value 0.08. This means that at large values of $p^+$, a condition corresponding to highly accelerating flow, the transitional region close to the wall will be larger. In this case, the skin-friction values will be lower than those obtained by $A^+ = 26$, corresponding to flat-plate flow. Figure 1b is a plot of Eq. (18) that shows the effect of $v_w^+$ on the damping constant for several $p^+$ values. The results indicate that the damping constant increases appreciably in accelerating flows (positive $p^+$) and with suction increases even more, as is to be expected. Figure 1c shows the effect of $p^+$ on the damping constant for several $v_w^+$-values. As is to be expected, the damping constant increases with suction and positive $p^+$.

† Here we assume that the sublayer Reynolds number for turbulent flows with pressure gradient and mass transfer does not change from its value for unblown flat-plate flows.

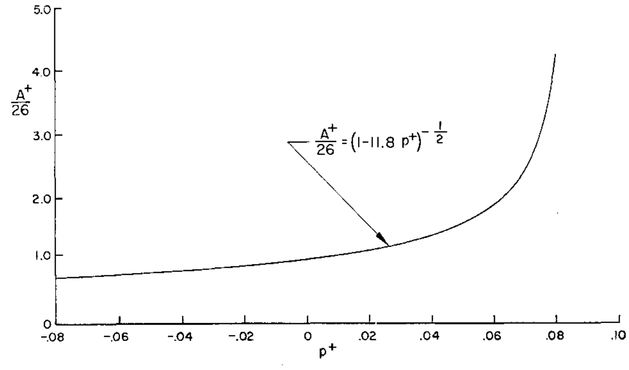



**Fig. 1a Deviation of damping constant of a pressure gradient flow from that of a nonporous flat-plate flow.**

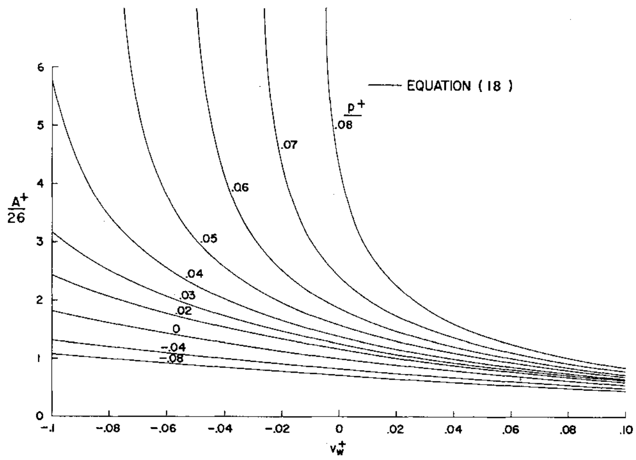



**Fig. 1b Deviation of damping constant with blowing parameter $v^+_w$ for several $p^+$ values.**

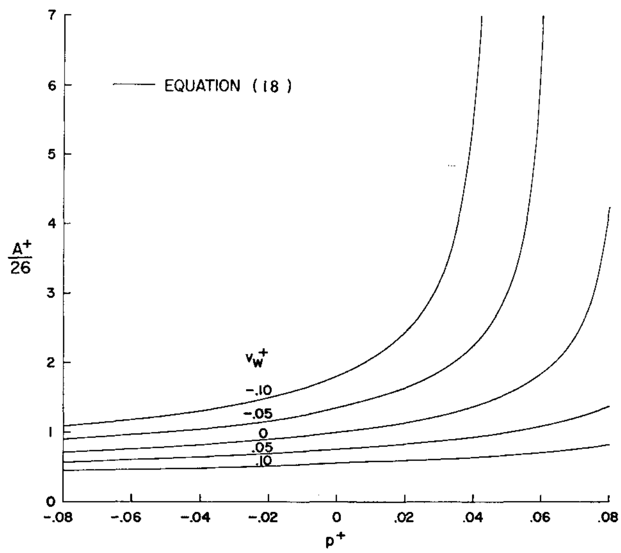



**Fig. 1c Deviation of damping constant with pressure gradient parameter $p^+$ for several $v^+_w$ values.**

## 3. Comparison With Experiment

Needless to say, the proposed modification of Van Driest's modified mixing-length expression is empirical, and like most expressions used in calculations of turbulent flows, it must be checked with experiment. This was done by comparing the calculated $A^+$ values from Eq. (18) with those obtained experimentally, and by using the eddy-viscosity method described in Ref. 2. In the latter case the boundary layer is re-

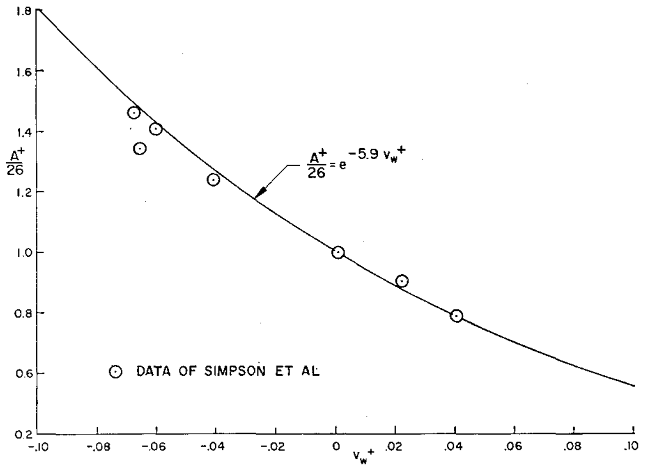



**Fig. 2 Deviation of damping constant of a porous flat-plate flow from that of a nonporous flat-plate flow.**

garded as a composite layer consisting of inner and outer regions for which separate expressions for eddy viscosity are used. In other words, the solution of Eq. (14) is obtained by eliminating the Reynolds shear-stress term by the following eddy-viscosity formulation:

$$-\rho\langle u'v'\rangle = \rho\epsilon\, \partial u/\partial y \tag{22}$$

where $\epsilon$ is given by

$$\epsilon = \begin{cases} \epsilon_i = (0.4y)^2\left[1 - \exp -\left(\frac{y^+}{A^+}\right)\right]^2 \left|\frac{\partial u}{\partial y}\right| \times \quad 0 \le y \le yc & (23a) \\ \epsilon_0 = 0.0168\left[\int_0^\infty (u_e - u)dy\right] \times \left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1} \quad y_c \le y \le \delta & (23b) \end{cases}$$

and the matching point $y_c$ between two regions is obtained from the continuity of eddy viscosity, namely, $\epsilon_i = \epsilon_0$. In Eq. (23a) the damping constant $A^+$ is given by Eq. (18).

Figures 2 and 3 show the deviation of damping constant $A^+$ with mass transfer for flat-plate flows. The experimental values of $A^+$ were obtained from the data of Simpson et al.[4] and Kendall,[5] and were reported in Ref. 6 by Bushnell and Beckwith. Figure 3 also shows the curve faired to the experimental data used by Bushnell and Beckwith. The calculations were made by using Eq. (21). The skin-friction values for Eq. (21) were obtained from Simpson's data[4] for blowing

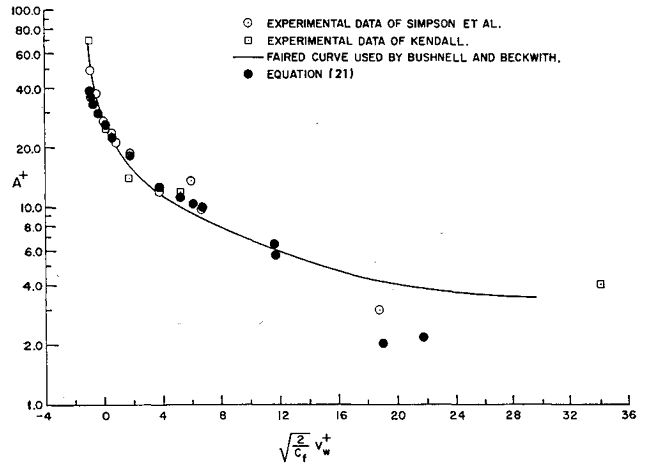



**Fig. 3 Comparison of calculated and experimental damping constants for a flat-plate flow with mass transfer.**

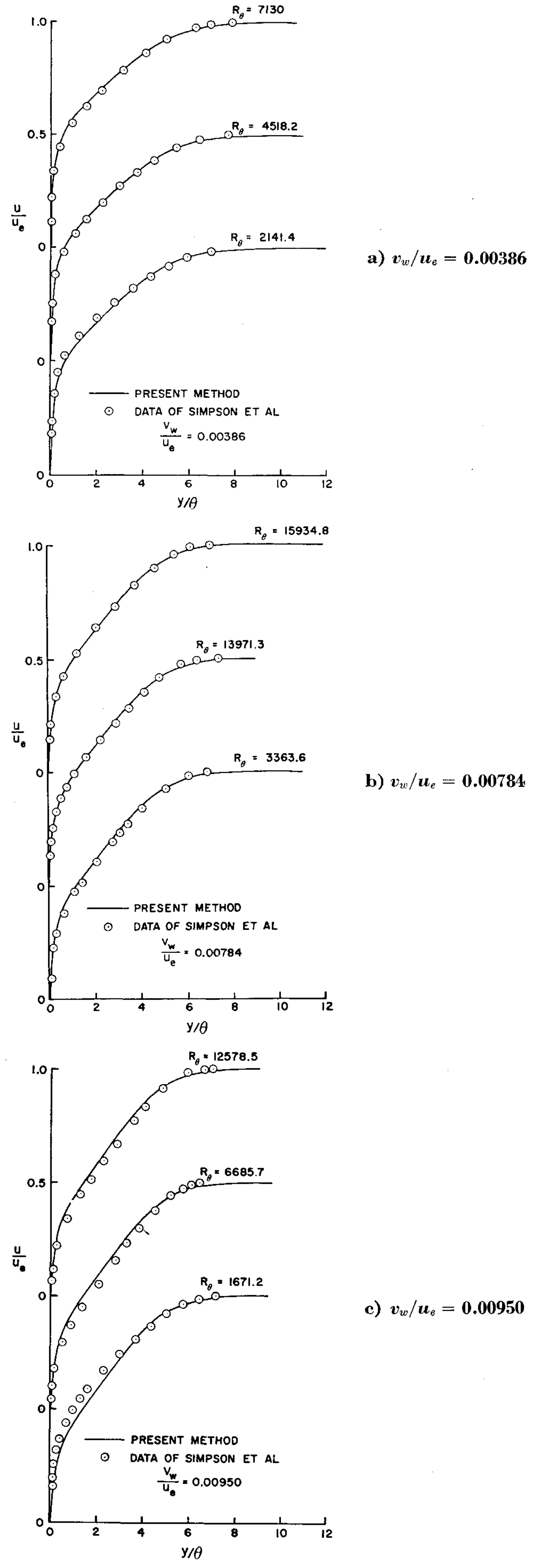
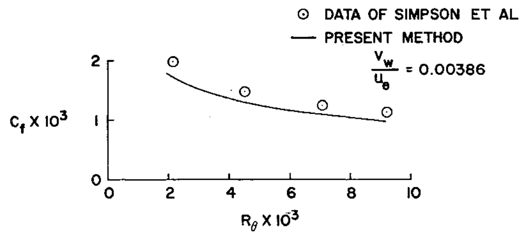
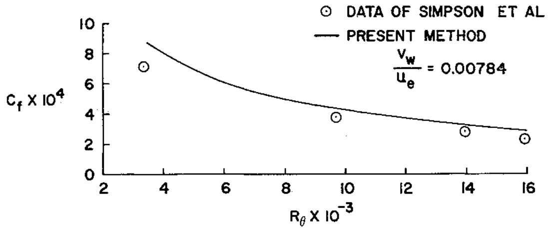



**Fig. 4 Comparison of calculated and experimental velocity profiles for the blown boundary layer measured by Simpson et al.**




a) $v_w/u_e = 0.00386$




b) $v_w/u_e = 0.00784$

**Fig. 5 Comparison of calculated and experimental skin-friction values for the blown boundary layer measured by Simpson et al.**[4]

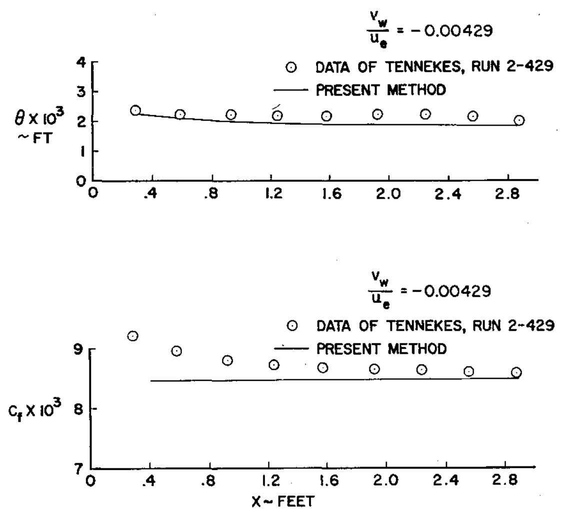



**Fig. 7 Comparison of calculated and experimental (top) momentum thickness values and (bottom) skin-friction values for the sucked boundary layer measured by Tennekes.**

and from Tennekes' data[7] for suction. The agreement is quite satisfactory. In Fig. 3 the agreement between Eq. (21), the experimental data, and the faired curve is very good for blowing parameters up to 14. For larger blowing parameters, the calculated $A^+$ values deviate from the faired curve used by Bushnell and Beckwith, but seem to agree reasonably well with experimental data except for one value.

Figures 4a, 4b, and 4c show the calculated velocity profiles obtained by the eddy-viscosity formulation given in Eq. (23) for blowing rates of $v_w/u_e = 0.00386$, 0.00784, and 0.00950, respectively, for the experimental data of Simpson et al.[4] In these calculations the damping constant $A^+$ as given by Eq. (21) was used. It is important to note that calculations could not be made with the unmodified damping constant given by $A^+ = 26$. Figures 5a and 5b show a comparison of calculated and experimental skin-friction values obtained for two blowing rates, $v_w/v_e = 0.00386$ and 0.00784, respectively. In general, the agreement seems satisfactory, and a slight discrepancy in skin-friction values could be caused by the procedure that was followed in making comparisons. Ideally, one should start the theoretical calculations by inputting the initial velocity profile obtained from the experimental data. However, in the present comparisons, an effective length that matched the momentum thickness at the station where blowing began was determined. This procedure does not necessarily match the initial skin-friction coefficient at that station. However, as the calculations continue in the streamwise direction, the initial discrepancy in the local skin friction decreases. For example, when the comparisons were made for the blowing ratio of $v_w/u_e = 0.00950$, the calculated and experimental skin-friction values were $5.5 \times 10^{-4}$ and $3.4 \times 10^{-4}$, respectively, at the station where the matching was made ($R_\theta = 4.3 \times 10^3$). Downstream, at sufficiently higher Reynolds number, the discrepancy became quite small. For example, at $R_\theta = 10^3$, the experimental and calculated local skin-friction values were $1 \times 10^{-4}$ and $1.1 \times 10^{-4}$, respectively, and at $R_\theta = 17 \times 10^3$, they were $1 \times 10^{-4}$ and $0.98 \times 10^{-4}$, respectively.

Comparisons were also made for the sucked boundary layers. The experimental data of Tennekes[7] were used for this purpose. Figure 6 shows a comparison of calculated and experimental velocity profiles for a suction rate of $v_w/u_e = -0.00429$. Figures 7a and 7b show comparisons of calculated and experimental values of momentum thickness and

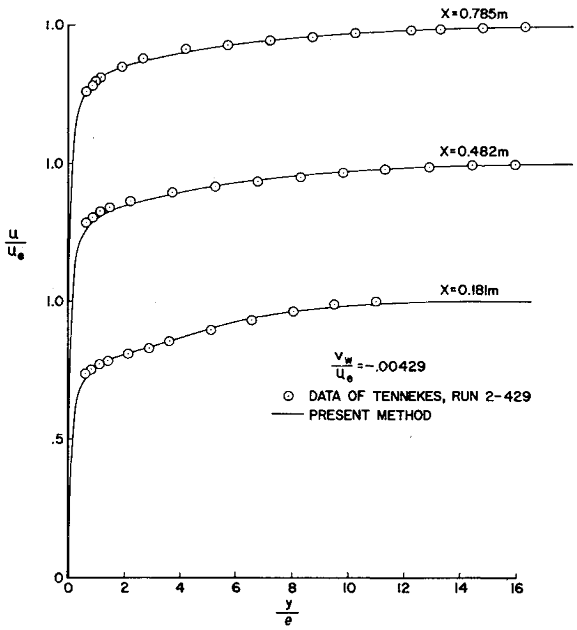



**Fig. 6 Comparison of calculated and experimental velocity profiles for the sucked boundary layer measured by Tennekes.**

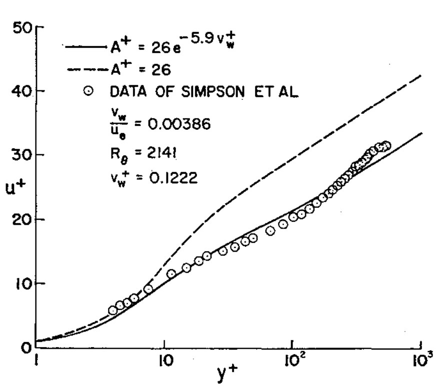



**Fig. 8 Comparison of calculated and experimental velocity profiles for the boundary layer measured by Simpson et al.**

local skin friction, respectively. In general, the agreement is quite satisfactory.

A comparison between results obtained by the old damping constant and by the new damping constant is also made for a flat-plate flow with mass transfer. This was done as follows. If we neglect the $\partial u/\partial x$ term in the continuity equation, it follows that the momentum equation can be written as

$$\rho v_w(du/dy) = d\tau/dy \tag{24}$$

where

$$\tau = \tau_l + \tau_t = \mu \frac{du}{dy} + \rho k^2 y^2 \left[1 - \exp\left(-\frac{y}{A}\right)\right]^2 \left(\frac{\partial u}{dy}\right)^2 \tag{25}$$

Integrating Eq. (24) and using Eq. (25), we can express the resulting expression, in dimensionless form, as

$$\frac{du^+}{dy^+} = \frac{2(v_w{}^+u^+ + 1)}{1 + \{1 + 4(v_w{}^+u^+ + 1)k^2(y^+)^2[1 - \exp(-y^+/A^+)]^2\}^{1/2}} \tag{26}$$

Figure 8 shows a comparison of calculated and experimental velocity profiles for a blowing rate of $v_w/u_e = 0.00386$, for the experimental data of Simpson et al.[4] These calculations were made by integrating Eq. (26) and by using the original damping constant, $A^+ = 26$, and the present damping constant, which in this case is the expression given by Eq. (21). The results show the marked improvement of the present formulation.

For the test cases considered in this paper, the calculated results indicate that the proposed modification of Van Driest's expression is quite useful and gives good agreement with experiment. Further comparisons on the same subject, which is reported in Ref. 8, also show the same good agreement and give further support to the proposed modification.

## 4. Concluding Remarks

Van Driest's theory, which provides a continuous velocity and shear distribution for turbulent flow near a nonporous wall, is extended to turbulent flow near a porous wall with pressure gradient. Several comparisons were made with the experimental data. The results show that the proposed modification of Van Driest's theory seems to be suitable for carrying the calculations right down to the wall to compute flows with pressure gradients and mass transfer. In the absence of a better empirical fit, the new formulation appears to be quite useful.

## References

[1] Van Driest, E. R., "On Turbulent Flow Near a Wall," *Journal of Aerospace Science*, Vol. 23, No. 11, Nov. 1956, pp. 1007–1011.

[2] Cebeci, T. and Smith, A. M. O., "A Finite-Difference Solution of the Incompressible Turbulent Boundary-Layer Equations by an Eddy-Viscosity Concept," *Computation of Turbulent Boundary Layers*, AFOSR-IFP-Stanford Conference, Vol. 1, 1968, Stanford Univ., pp. 346–355.

[3] Patankar, S. V. and Spalding, D. B., *Heat and Mass Transfer in Boundary Layers*, Morgan-Grompian, London, 1967.

[4] Simpson, R. L., Kays, W. M., and Moffat, R. J., "The Turbulent Boundary Layer on a Porous Plate: An Experimental Study of the Fluid Dynamics with Injection and Suction," Rept. HMT-2, 1967, Stanford Univ.

[5] Kendall, R. M. et al., "Mass Momentum and Heat Transfer Within a Turbulent Boundary Layer with Foreign Gas Mass Transfer at the Surface," Rept. 111, Feb. 1964, Vidya, Division of Itek Corp.

[6] Bushnell, D. M. and Beckwith, I. E., "Calculation of Nonequilibrium Hypersonic Turbulent Boundary Layers and Comparisons with Experimental Data," AIAA Paper 69-584, San Francisco, Calif., 1969.

[7] Tennekes, H., "Similarity Laws for Turbulent Boundary Layers with Suction or Injection," Rept. VTH-119, Dec. 1969, Technological University, Delft.

[8] Cebeci, T. and Mosinskis, G., "Calculation of Heat and Mass Transfer in Turbulent Flows at Low Mach Numbers," Rept. DAC 70015, Douglas Aircraft Co. 1969, Long Beach, Calif.